

# THE BITWISTED CARTESIAN MODEL FOR THE FREE LOOP FIBRATION

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**ABSTRACT.** Using the notion of truncating twisting function from a simplicial set to a cubical set a special, bitwisted, Cartesian product of these sets is defined. For the universal truncating twisting function, the (co)chain complex of the corresponding bitwisted Cartesian product agrees with the standard Cartier (Hochschild) chain complex of the simplicial (co)chains. The modelling polytopes  $F_n$  are constructed. An explicit diagonal on  $F_n$  is defined and a multiplicative model for the free loop fibration  $\Omega Y \rightarrow \Lambda Y \rightarrow Y$  is obtained. As an application we establish an algebra isomorphism  $H^*(\Lambda Y; \mathbb{Z}) \approx S(U) \otimes \Lambda(s^{-1}U)$  for the polynomial cohomology algebra  $H^*(Y; \mathbb{Z}) = S(U)$ .

## 1. INTRODUCTION

Let  $\Lambda Y$  denote the free loop space of a topological space  $Y$ , i.e. the space of all continuous maps from the circle  $S^1$  into  $Y$ , and let  $\Omega Y \rightarrow \Lambda Y \xrightarrow{\xi} Y$  be the free loop fibration. Since  $\xi$  can be viewed as obtained from the path fibration  $\Omega Y \rightarrow PY \rightarrow Y$  by means of the conjugation action  $ad : \Omega Y \times \Omega Y \rightarrow \Omega Y$  [12], one could apply [9] to construct for  $\xi$  the twisted tensor product  $C_*(Y) \otimes_{\tau_*} \Omega C^*(Y)$ . However, the induced action  $ad_* : \Omega C_*(Y) \otimes \Omega C_*(Y) \rightarrow \Omega C_*(Y)$  is hardly to write down by explicit formulas; an alternative way for modelling  $\Lambda Y$  is the Cartier chain complex  $\Lambda C_*(Y)$  of the singular simplicial chain coalgebra  $C_*(Y)$  thought of as a specific, *bitwisted*, tensor product  $\Lambda C_*(Y) = C_*(Y) \otimes_{\tau_* \otimes \tau_*} \Omega C_*(Y)$  [5], [12]. Accordingly, for a twisting truncating function  $\tau : X \rightarrow Q$  from a 1-reduced simplicial set to a monoidal cubical set  $Q$ , we modify the twisted Cartesian product  $X \times_{\tau} Q$  from [9] to obtain a *bitwisted Cartesian product*  $X \times_{\tau} Q$  such that  $\Lambda C_*(X) = C_*^{\odot}(X \times_{\tau} Q)$  whenever  $Q = \Omega X$ , a monoidal cubical set, and  $\tau = \tau_U : X \rightarrow \Omega X$ , the universal truncating twisting function, constructed in [9]. Dually, for the Hochschild complex  $\Lambda C^*(X)$  of the simplicial cochain algebra  $C^*(X)$  we get the inclusion of cochain complexes  $\Lambda C^*(X) \subset C_{\odot}^*(X \times_{\tau} \Omega X)$  (here we have equality when the graded sets have finite type). The required model for  $\Lambda Y$  is obtained by taking  $X = \text{Sing}^1 Y$ , the Eilenberg 1-subcomplex of the singular simplicial set  $\text{Sing } Y$ .

We construct polytopes  $F_n$  referred to as *freehedra* and introduce the notion of an  $F_n$ -set. The motivation is that the bitwisted Cartesian product  $X \times_{\tau} \Omega X$  above admits such a combinatorial structure in a canonical way. As the standard simplices  $\Delta^n$ ,  $n \geq 0$ , are the modelling polytopes for a simplicial set, the Cartesian

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products  $F_i \times I^j$ ,  $i, j \geq 0$ , with  $I^j$  the standard  $j$ -cube serve as modelling polytopes for an  $F_n$ -set. A universal example for an  $F_n$ -set is the *singular* complex  $\text{Sing}^F Y$ , obtained as the set of continuous maps  $I^k \times F_m \times I^\ell \rightarrow Y$  for  $k, \ell, m \geq 0$ . The normalized (co)chain complex  $C_*^\odot(Y)$  ( $C_*^\odot(Y)$ ) of  $\text{Sing}^F Y$  also gives the singular (co)homology of  $Y$ . In particular, we construct a map of  $F_n$ -sets

$$\Upsilon : \text{Sing}^1 Y_{\tau \times \tau} \Omega \text{Sing}^1 Y \rightarrow \text{Sing}^F \Lambda Y$$

that extends the cubical map  $\omega : \Omega \text{Sing}^1 Y \rightarrow \text{Sing}^I \Omega Y$  realizing Adams' cobar equivalence  $\omega_* : \Omega C_*(Y) \rightarrow C_*^\square(\Omega Y)$  [1]. The image  $\Upsilon(\sigma_m, \sigma'_n)$  for  $m, n > 1$  consists of singular  $(F_m \times I^n)$ -polytopes that are purely determined by a choice of measuring homotopy for the *commutativity* of the standard loop product of  $\Omega Y$  by the inclusion  $\Omega Y \subset \Lambda Y$ . Such a relationship is possible since  $F_m$  admits a representation as an explicit subdivision of the cube  $I^m = I^{m-1} \times I$ .

We construct an explicit diagonal  $\Delta_F$  for  $F_n$ . In a standard way this diagonal together with the Serre diagonal of the cubes yields the diagonal of an arbitrary  $F_n$ -set. Consequently, the  $F_n$ -set structure of the bitwisted Cartesian product  $\text{Sing}^1 Y_{\tau \times \tau} \Omega \text{Sing}^1 Y$ , as a by-product, determines a comultiplication on the Cartier complex  $\Lambda C_*(Y)$ . Dually, we obtain a multiplication on the Hochschild complex  $\Lambda C^*(Y)$ . More precisely, a combinatorial analysis of  $\Delta_F$  shows that the multiplication on the Hochschild complex  $\Lambda C^*(Y)$  involves a canonical homotopy  $G$ -algebra (hga) structure on the simplicial cochain algebra  $C^*(Y)$  [3], [6], [7] (compare [9]). Thus, applying the cohomology functor for  $\Upsilon$  we obtain a natural algebra isomorphism

$$\Upsilon^* : HH_*(C^*(Y; \mathbb{k})) \xleftarrow{\approx} H^*(\Lambda Y; \mathbb{k})$$

in which on the left-hand side the product is given by formula (6.5) and  $\mathbb{k}$  is a commutative ring with identity. Note also that the multiplication on the Hochschild complex  $\Lambda C^*(Y)$  is not associative, but it can be extended to an  $A_\infty$ -algebra structure [20].

As an application of the above algebra isomorphism we establish the fact that the standard shuffle product on the Hochschild complex  $\Lambda H$  with  $H = H^*(Y; \mathbb{k})$  polynomial is *geometric*. More precisely, given a cocycle  $z \in C^n(Y; \mathbb{k})$ , the product  $z \smile_1 z \in C^{2n-1}(Y; \mathbb{k})$  is a cocycle for  $n$  even or for  $n$  odd too, when  $z^2$  is of the second order. In any case, the class  $[z \smile_1 z] \in H^{2n-1}(Y; \mathbb{k})$  is defined and is denoted by  $Sq_1[z]$ . Clearly, if  $H = H^*(Y; \mathbb{k})$  is a polynomial algebra and  $\mathbb{k}$  has no 2-torsion, then  $H$  is evenly graded and  $Sq_1$  is identically zero on  $H$ . However,  $H$  may have odd dimensional generators for  $\mathbb{k} = \mathbb{Z}/2\mathbb{Z}$ . We have the following theorem that generalizes a well-known result when  $\mathbb{k}$  is a field.

**Theorem 1.** *Let  $H = H^*(Y; \mathbb{k}) = S(U)$  be a polynomial algebra such that  $Sq_1 = 0$  on  $H$ . Let  $\Lambda(s^{-1}U)$  be the exterior algebra over the desuspension of the polynomial generators  $U$ . Then there are algebra isomorphisms*

$$H(\Lambda Y; \mathbb{k}) \approx S(U) \otimes \Lambda(s^{-1}U) \approx H(Y; \mathbb{k}) \otimes H(\Omega Y; \mathbb{k}).$$

Note that in an ungraded setting of  $H = H(Y; \mathbb{k})$  the middle term above is interpreted as the module of differential forms denoted by  $\Omega_{H|\mathbb{k}}$  [10]. Using a *strong homotopy commutative* structure of  $C^*(Y; \mathbb{k})$  the multiplication on  $\Lambda C^*(Y; \mathbb{k})$  is constructed in [14] for  $\mathbb{k}$  to be a field (see also [13] for references). Though there is a close relationship between the strong homotopy commutative and hga structures, the explicit formula for the product on the Hochschild chain complex  $\Lambda C^*(Y)$  in

terms of the hga operations is heavily used here (see Lemma 1). Moreover, the proof of the above theorem suggests further calculations of the Hochschild homology for a more wide class of spaces having, for example, the cohomology isomorphic to smooth algebras.

Restricted to finitely generated polynomial algebras, i.e.,  $U$  is finite dimensional, Theorem 1 agrees with the solution of the Steenrod problem given in [2]. At the end of the paper we include an example showing that there is a tensor product algebra  $C$  but with perturbed differential such that the  $E_\infty$ -term of the spectral sequence of  $C$  is isomorphic as algebras with the  $E_\infty$ -term of the Serre spectral sequence of the free loop fibration with the base  $Y = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ , nevertheless  $H^*(C)$  differs from  $H^*(\Lambda Y)$  as algebras.

Finally, note that a large part of the paper consists of an editing excerpt from the earlier preprint [17] where in particular the polytopes  $F_n$  were constructed. In the meantime 3-dimensional polytope  $F_3$  appeared in [4] (see also [11]).

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## 2. SOME PRELIMINARIES AND CONVENTIONS

We adopt the notions and notations from [9].

**2.1. Cobar and Bar constructions.** Let  $\mathbb{k}$  be a commutative ring with identity. For a  $\mathbb{k}$ -module  $M$ , let  $T(M)$  be the tensor algebra of  $M$ , i.e.  $T(M) = \bigoplus_{i=0}^\infty M^{\otimes i}$ . An element  $a_1 \otimes \dots \otimes a_n \in M^{\otimes n}$  is denoted by  $[a_1] \cdots [a_n]$ . We denote by  $s^{-1}M$  the desuspension of  $M$ , i.e.  $(s^{-1}M)_i = M_{i+1}$ .

Let  $(C, d_C, \Delta)$  be a 1-reduced dgc, i.e.  $C_0 = \mathbb{k}$ ,  $C_1 = 0$ . Denote  $\bar{C} = s^{-1}(C_{>0})$ . Let  $\Delta = Id \otimes 1 + 1 \otimes Id + \Delta'$ . The (reduced) cobar construction  $\Omega C$  on  $C$  is the tensor algebra  $T(\bar{C})$ , with differential  $d = d_1 + d_2$  defined for  $\bar{c} \in \bar{C}_{>0}$  by

$$d_1[\bar{c}] = - \left[ \overline{d_C(c)} \right] \quad \text{and} \quad d_2[\bar{c}] = \sum (-1)^{|c'|} [\bar{c}' | \bar{c}''], \quad \text{for} \quad \Delta'(c) = \sum c' \otimes c'',$$

extended as a derivation. The acyclic cobar construction  $\Omega(C; C)$  is the twisted tensor product  $C \otimes \Omega C$  in which the tensor differential is twisted by the universal twisting cochain  $\tau_* : C \rightarrow \Omega C$  being an inclusion of degree  $-1$ .

Let  $(A, d_A, \mu)$  be a 1-reduced dga. The (reduced) bar construction  $BA$  on  $A$  is the tensor coalgebra  $T(\bar{A})$ ,  $\bar{A} = s^{-1}(A_{>0})$ , with differential  $d = d_1 + d_2$  given for  $[\bar{a}_1] \cdots [\bar{a}_n] \in T^n(\bar{A})$  by

$$d_1[\bar{a}_1] \cdots [\bar{a}_n] = - \sum_{i=1}^n (-1)^{\epsilon_{i-1}} [\bar{a}_1] \cdots [\overline{d_A(a_i)}] \cdots [\bar{a}_n],$$

and

$$d_2[\bar{a}_1] \cdots [\bar{a}_n] = - \sum_{i=1}^{n-1} (-1)^{\epsilon_i} [\bar{a}_1] \cdots [\overline{\bar{a}_i \bar{a}_{i+1}}] \cdots [\bar{a}_n],$$

where  $\epsilon_i = \epsilon_i^a = |a_1| + \dots + |a_i| + i$ . The acyclic bar construction  $B(A; A)$  is the twisted tensor product  $A \otimes BA$  in which the tensor differential is twisted by the universal twisting cochain  $\tau^* : BA \rightarrow A$  being a projection of degree 1.

**2.2. Cartier and Hochschild chain complexes.** Let  $(C, d_C, \Delta)$  be a 1-reduced dgc and let  $\Delta = Id \otimes 1 + \Delta_1 = 1 \otimes Id + \Delta_2$ . The (normalized) *Cartier complex*  $\Lambda C$  of  $C$  [5] is  $C \otimes \Omega C$  with differential  $d$  defined by  $d = d_C \otimes 1 + 1 \otimes d_{\Omega C} + \theta_1 + \theta_2$ , where

$$\begin{aligned}\theta_1(v \otimes [\bar{c}_1 | \cdots | \bar{c}_n]) &= -\sum (-1)^{|v'_1|} v'_1 \otimes [\bar{v}_1'' | \bar{c}_1 | \cdots | \bar{c}_n], \\ \theta_2(v \otimes [\bar{c}_1 | \cdots | \bar{c}_n]) &= \sum (-1)^{(|v'_2|+1)(|v'_2|+\epsilon_n)} v'_2 \otimes [\bar{c}_1 | \cdots | \bar{c}_n | \bar{v}_2'], \\ \Delta_i(v) &= \sum v'_i \otimes v''_i, \quad i = 1, 2.\end{aligned}$$

The homology of  $\Lambda C$  is called the Cartier homology of a dgc  $C$  and is denoted by  $HH_*(C)$ .

Note that the components  $\theta_1$  and  $\theta_2$  above can be thought of as obtained by applying the universal twisting cochain  $\tau_* : C \rightarrow \Omega C$  on the tensor product  $C \otimes \Omega C$  twice:

$$C \otimes \Omega C \xrightarrow{\Delta \otimes 1} C \otimes C \otimes \Omega C \xrightarrow{1 \otimes \tau_* \otimes 1 + (1 \otimes 1 \otimes \tau_*)^T} C \otimes \Omega C \otimes \Omega C \xrightarrow{1 \otimes \mu} C \otimes \Omega C,$$

where  $T : C \otimes (C \otimes \Omega C) \rightarrow (C \otimes \Omega C) \otimes C$ ; consequently,  $\Lambda C$  is a *bitwisted tensor product*  $C_{\tau_* \otimes \tau_*} \Omega C$ .

The (normalized) *Hochschild complex*  $\Lambda A$  of a 1-reduced associative dg algebra  $A$  ([10]) is  $A \otimes BA$  with differential  $d$  defined by  $d = d_A \otimes 1 + 1 \otimes d_{BA} + \theta^1 + \theta^2$ , where

$$\begin{aligned}\theta^1(u \otimes [\bar{a}_1 | \cdots | \bar{a}_n]) &= -(-1)^{|u|} u a_1 \otimes [\bar{a}_2 | \cdots | \bar{a}_n], \\ \theta^2(u \otimes [\bar{a}_1 | \cdots | \bar{a}_n]) &= (-1)^{(|a_n|+1)(|u|+\epsilon_{n-1}^a)} a_n u \otimes [\bar{a}_1 | \cdots | \bar{a}_{n-1}].\end{aligned}$$

The homology of  $\Lambda A$  is called the Hochschild homology of a dga  $A$  and is denoted by  $HH_*(A)$ .

Dually, the components  $\theta^1$  and  $\theta^2$  can be thought of as obtained by applying the universal twisting cochain  $\tau^* : BA \rightarrow A$  on the tensor product  $A \otimes BA$  twice; consequently,  $\Lambda A$  is a *bitwisted tensor product*  $A_{\tau^* \otimes \tau^*} BA$ .

### 3. THE POLYTOPES $F_n$

It is well known that the standard cube  $I^n$  can be viewed as obtained from the standard simplex  $\Delta^n$  with vertices  $(v_0, v_1, \dots, v_n)$  by a truncating procedure starting either at the minimal vertex  $v_0$  or at the maximal vertex  $v_n$ ; One gets a polytope, called a freehedron and denoted by  $F_n$ , when the truncations start at both vertices  $v_0$  and  $v_n$  simultaneously; since these two truncations do not meet, we can begin the truncation procedure at vertex  $v_0$  to obtain first the standard cube  $I^n$ , and then continue the same procedure at vertex  $v_n$  to recover  $F_n$ . Thus, we obtain the canonical cellular projection, a "healing" map,  $\varphi : F_n \rightarrow \Delta^n$  such that it factors through the projections  $\phi : F_n \rightarrow I^n$  and  $\psi : I^n \rightarrow \Delta^n$  (see Figures 1-3).

It is convenient, to regard  $F_n$  as a subdivision of  $I^n$  and to give the following inductive construction of  $F_n$  whose faces are labelled three types of face operators  $d_i^0, d_i^1$  and  $d_i^2$  with  $d_1^1 = d_1^2$ . Let  $F_0$  and  $F_1$  be a point and an interval respectively. If  $F_{n-1}$  has been constructed, let  $e_i^\epsilon$  denote the face  $(x_1, \dots, x_i, \epsilon, x_{i+1}, \dots, x_{n-1}) \subset I^n$  where  $\epsilon = 0, 1$  and  $1 \leq i \leq n$ . Then  $F_n$  is the subdivision of  $F_{n-1} \times I$  given below

and its various  $(n-1)$ -faces are labelled as indicated:

Face of $F_n$	Face operator
$e_i^0$	$d_i^0, \quad 1 \leq i \leq n$
$e_i^1$	$d_i^1, \quad 2 \leq i \leq n$
$d_i^2 \times I$	$d_i^2, \quad 1 \leq i \leq n-2$
$d_{n-1}^2 \times [0, \frac{1}{2}]$	$d_{n-1}^2,$
$d_n^2 \times [\frac{1}{2}, 1]$	$d_n^2,$

Figure 1:  $F_n$  as a subdivision of  $F_{n-1} \times I$  for  $n = 2, 3$ .

Thus,  $F_2$  is a pentagon,  $F_3$  has eight 2-faces (4 pentagon and 4 quadrilateral), 18 edges and 12 vertices (compare [4]). In particular, the sequence of codimension 1 faces of  $F_n$ ,  $n \geq 1$ , is an arithmetic progression with difference 3.

**3.1. The singular  $F_n$ -set.** Before we give the notion of an *abstract  $F_n$ -set* below, let consider its universal example, the *singular  $F_n$ -set*  $\text{Sing}^F Y$  of a topological space  $Y$ , i.e. the set of all continuous maps  $\{I^k \times F_m \times I^\ell \rightarrow Y\}_{m,k,\ell \geq 0}$ . The face and degeneracy operators are defined as follows.

Obviously, the face  $d_i^1(F_m)$  is homeomorphic to  $F_{m-1}$  for each  $i$ , while according to the orientation of the cube  $I^m$  the faces  $d_i^0(F_m)$  and  $d_i^2(F_m)$  are homeomorphic to  $F_{i-1} \times I^{m-i}$  and  $I^{i-1} \times F_{m-i}$  respectively; let these homeomorphisms be realized by the following inclusions

$$\begin{aligned} \bar{\delta}_i^0 &: F_{i-1} \times I^{m-i} \hookrightarrow F_m, \\ \bar{\delta}_i^1 &: F_{m-1} \hookrightarrow F_m, \\ \bar{\delta}_i^2 &: I^{i-1} \times F_{m-i} \hookrightarrow F_m \end{aligned}$$

with  $\bar{\delta}_1^2 = \bar{\delta}_1^1$ . Let  $\iota_i^\epsilon : I^k \hookrightarrow I^{k+1}$  be the inclusion defined by  $\iota_i^\epsilon(I^k) = e_i^\epsilon$ . Given  $k, \ell, m \geq 0$  with  $m + \ell = r_1$ ,  $k + m + \ell = r_2$ , define the inclusions

$$\delta_i^\epsilon : I^{k_{\epsilon,i}} \times F_{m_{\epsilon,i}} \times I^{\ell_{\epsilon,i}} \hookrightarrow I^k \times F_m \times I^\ell$$

by

$$\delta_i^\epsilon = \begin{cases} 1 \times \bar{\delta}_i^0 \times 1, & \epsilon = 0, & (k, m, \ell)_{0,i} = (k, m-i, \ell+i-1), & 1 \leq i \leq m, \\ 1 \times \bar{\delta}_i^1 \times 1, & \epsilon = 1, & (k, m, \ell)_{1,i} = (k, m-1, \ell), & 1 \leq i \leq m, \\ 1 \times 1 \times \iota_{i-m}^\epsilon, & \epsilon = 0, 1, & (k, m, \ell)_{\epsilon,i} = (k, m, \ell-1), & m < i \leq r_1, \\ \iota_{i-r_1}^\epsilon \times 1 \times 1, & \epsilon = 0, 1, & (k, m, \ell)_{\epsilon,i} = (k-1, m, \ell), & r_1 < i \leq r_2, \\ 1 \times \bar{\delta}_i^2 \times 1, & \epsilon = 2, & (k, m, \ell)_{2,i} = (k+i-1, m-i, \ell), & 1 \leq i \leq m. \end{cases}$$

Then define the face operators

$$d_i^\epsilon : (\text{Sing}^F Y)^{m,n} \rightarrow (\text{Sing}^F Y)^{m_{\epsilon,i}, n_{\epsilon,i}}$$

for  $f \in (\text{Sing}^F Y)^{m,n}$ ,  $f : I^k \times F_m \times I^\ell \rightarrow Y$ ,  $n = k + \ell$ , by  $d_i^\epsilon(f) = f \circ \delta_i^\epsilon$ ,  $\epsilon = 0, 1, 2$ .

Given  $1 \leq i \leq n + 1$ , define the degeneracy operators

$$\eta_i : (\text{Sing}^F Y)^{m,n} \rightarrow (\text{Sing}^F Y)^{m,n+1}$$

by

$$\eta_i(f) = \begin{cases} f \circ (1 \times 1 \times \bar{\eta}_i), & 1 \leq i \leq \ell + 1 \\ f \circ (\bar{\eta}_{i-\ell} \times 1 \times 1), & \ell + 1 < i \leq n + 1, \quad n = k + \ell, \end{cases}$$

with  $\bar{\eta}_i : I^{r+1} \rightarrow I^r$ ,  $\bar{\eta}_i(x_1, \dots, x_{r+1}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{r+1})$ .

Thus, we obtain the singular  $F_n$ -set

$$\{\text{Sing}^F Y = \bigcup_{k,m,\ell \geq 0} [(\text{Sing}^F Y)^{m,k+\ell} = \{I^k \times F_m \times I^\ell \rightarrow Y\}], d_i^0, d_i^1, d_i^2, \eta_i\}$$

the face and degeneracy operators of which satisfy the following equalities:

$$(3.1) \quad \begin{aligned} d_i^0 d_j^0 &= d_{j-1}^0 d_i^0, & i < j, \\ d_i^1 d_j^1 &= d_{j-1}^1 d_i^1, & i < j \text{ and } (i, j) \neq (1, 2) \text{ for } m > 0, \\ d_i^1 d_j^0 &= \begin{cases} d_{j-1}^0 d_i^1, & i < j, \\ d_j^0 d_{i+1}^1, & i \geq j, \end{cases} \\ d_i^2 d_j^0 &= d_{j-i}^0 d_i^2, \\ d_i^2 d_j^1 &= \begin{cases} d_{j-i}^1 d_i^2, & i < j - 1, \\ d_{m+n+j-i-2}^2 d_{i+1}^2, & i \geq j - 1 > 0, \end{cases} \\ d_i^2 d_j^2 &= d_{m+n-i}^0 d_{i+j}^2, \\ d_i^\epsilon \eta_j &= \begin{cases} \eta_{j-1} d_i^\epsilon & i < m + j, \\ 1 & i = m + j, \\ \eta_j d_{i-1}^\epsilon & i > m + j, \quad \epsilon = 0, 1, \end{cases} \\ d_i^2 \eta_j &= \eta_j d_i^2, \\ \eta_i \eta_j &= \eta_{j+1} \eta_i, & i \leq j. \end{aligned}$$

Note that the compositions  $d_1^1 d_2^1$  and  $d_1^1 d_1^1$  that are eliminated from the second equality above are involved in the fifth and sixth ones as  $d_1^2 d_2^1 = d_{m+n-1}^2 d_2^2$  and  $d_1^2 d_1^2 = d_{m+n-1}^2 d_2^2$  by taking into account that  $d_1^1 = d_1^2$ .

**3.2. Abstract freehedral sets.** For a general theory of polyhedral sets, see for example [15], [8]. Motivated by the underlying combinatorial structure of the bitwisted Cartesian product  $\mathbf{A}X$  below we give the following

**Definition 1.** An  $F_n$ -set  $\mathcal{CH}$  is a bigraded set  $\mathcal{CH} = \{\mathcal{CH}^{m,n}, m, n \geq 0\}$  with total grading  $m + n$  and three types of faces operators  $d_i^0, d_i^1, d_i^2$  with  $d_1^1 = d_1^2$  for  $m > 0$ ,

$$\begin{aligned} d_i^0 : \mathcal{CH}^{m,n} &\rightarrow \mathcal{CH}^{i-1, m+n-i}, & 1 \leq i \leq m, \\ d_i^0 : \mathcal{CH}^{m,n} &\rightarrow \mathcal{CH}^{m, n-1}, & m < i \leq m + n, \\ d_i^1 : \mathcal{CH}^{m,n} &\rightarrow \mathcal{CH}^{m-1, n}, & 1 \leq i \leq m, \\ d_i^1 : \mathcal{CH}^{m,n} &\rightarrow \mathcal{CH}^{m, n-1}, & m < i \leq m + n, \\ d_i^2 : \mathcal{CH}^{m,n} &\rightarrow \mathcal{CH}^{m-i, n+i-1}, & 1 \leq i \leq m, \end{aligned}$$

and degeneracy operators

$$\eta_i : \mathcal{CH}^{m,n} \rightarrow \mathcal{CH}^{m,n+1}, \quad 1 \leq i \leq n+1$$

satisfying structural identities (3.1).

A morphism of  $F_n$ -sets is a family of maps  $f = \{f_{m,n}\}$ ,  $f_{m,n} : \mathcal{CH}^{m,n} \rightarrow \mathcal{CH}'^{m,n}$ , commuting with all face and degeneracy operators.

Note that unlike  $d_i^0$  and  $d_i^1$  the face operator  $d_i^2$  is defined only for  $1 \leq i \leq m$ . Consequently, for  $m = 0$  we have only of two types of face operators  $d_i^\epsilon$ ,  $\epsilon = 0, 1$ , acting on  $\mathcal{CH}^{0,n}$  that satisfy the standard *cubical* relations; so that, in every  $F_n$ -set the subset  $\{\mathcal{CH}^{0,r}\}_{r \geq 0}$  together with the operators  $d^0$ ,  $d^1$  and  $\eta_i$  forms a cubical set. Furthermore, the operators  $d_1^0$  and  $d_m^2$  have the image in this subset.

The  $F_n$ -set structural relations also can be conveniently verified by means of the following combinatorics of the polytopes  $F_m$  (compare with Proposition 3.2 in [9]). The top dimensional cell of  $F_m$  is identified with the set  $0, 1, \dots, m$ , while any proper  $q$ -face  $u$  of  $F_m$  is expressed as (see Figure 2)

$$u = i_{s_t}, \dots, i_{s_{t+1}} [i_{s_{t+1}}, \dots, i_{s_{t+2}}] \dots [i_{s_{k-1}}, \dots, i_{s_k}, m] [0, i_1, \dots, i_{s_1}] [i_{s_1}, \dots, i_{s_2}] \dots [i_{s_{t-1}}, \dots, i_{s_t}], \\ 0 < i_1 < \dots < i_{s_t} < \dots < i_{s_k} < m, \quad q = s_k - k + 1.$$

where for  $t = 0$  the face  $u$  is assumed to have the form

$$u = i_1, \dots, i_{s_1} [i_{s_1}, \dots, i_{s_2}] \dots [i_{s_{k-1}}, \dots, i_{s_k}, m]$$

with  $0 \leq i_1 < \dots < i_{s_k} < m$ .

**Proposition 1.** *Let the face operators  $d^0, d^1, d^2$  act on  $a_0, a_1, \dots, a_m [b_0, \dots, b_{n+1}]$  (thought of as the top cell of  $F_m \times I^n$ ) by*

$$a_0, a_1, \dots, a_m [b_0, \dots, b_{n+1}] \xrightarrow{d_i^0} \begin{cases} a_0, a_1, \dots, a_{i-1} [a_{i-1}, \dots, a_m] [b_0, \dots, b_{n+1}], & 1 \leq i \leq m, \\ a_0, a_1, \dots, a_m [b_0, \dots, b_j] [b_j, \dots, b_{n+1}], & i = m+j \end{cases} \\ a_0, a_1, \dots, a_m [b_0, \dots, b_{n+1}] \xrightarrow{d_i^1} \begin{cases} a_1, \dots, a_m [b_0, \dots, b_{n+1}] [a_0, a_1], & i = 1, \\ a_0, a_1, \dots, \hat{a}_{i-1}, \dots, a_m [b_0, \dots, b_{n+1}], & 2 \leq i \leq m, \\ a_0, a_1, \dots, a_m [b_0, b_1, \dots, \hat{b}_j, \dots, b_{n+1}], & i = m+j \end{cases} \\ a_0, a_1, \dots, a_m [b_0, \dots, b_{n+1}] \xrightarrow{d_i^2} a_i, \dots, a_m [b_0, \dots, b_{n+1}] [a_0, a_1, \dots, a_i], \quad 1 \leq i \leq m.$$

Then the relations among  $d^\epsilon$ 's for  $\epsilon = 0, 1, 2$  agree with the  $F_n$ -set identities.

*Proof.* It is straightforward. □

A degeneracy operator  $\eta_i$  is thought of as adding a formal element  $*$  to the set  $a_0, a_1, \dots, a_m [b_0, b_1, \dots, b_{n+1}]$  at the  $(m+i+2)^{st}$  place:

$$a_0, a_1, \dots, a_m [b_0, b_1, \dots, b_{n+1}] \xrightarrow{\eta_i} a_0, a_1, \dots, a_m [b_0, b_1, \dots, b_i, *, b_{i+1}, \dots, b_{n+1}]$$

with the convention  $[b_0, b_1, \dots, b_i, *][*, b_{i+1}, \dots, b_{n+1}] = [b_0, b_1, \dots, b_{n+1}]$  that guarantees the equality  $d_{m+i}^0 \eta_i = Id = d_{m+i}^1 \eta_i$ .



Figure 2. The combinatorial description of freehedra  $F_2 = 012]$  and  $F_3 = 0123]$ .

#### 4. A DIAGONAL ON THE FREEHEDRA $F_n$

For the freehedron  $F_n$  define its integral chain complex  $(C_*(F_n), d)$  with differential

$$(4.1) \quad d = \sum_{i=1}^n (-1)^i (d_i^0 - d_i^1) + \sum_{i=2}^n (-1)^{(i-1)n} d_i^2.$$

Now define the map

$$\Delta_F : C_*(F_n) \rightarrow C_*(F_n) \otimes C_*(F_n),$$

for  $u_n \in C_n(F_n)$ ,  $n \geq 0$ , by

$$(4.2) \quad \Delta_F(u_n) = \sum_{(K,L)} \text{sgn}(K, L) d_{j_p}^0 \dots d_{j_1}^0(u_n) \otimes d_{i_q}^1 \dots d_{i_1}^1(u_n) +$$

$$\sum_{\substack{(K', L') \\ r+1 \leq i_q \leq p+1 \\ 0 \leq r < n}} (-1)^{j_{(r)} + r + (p+1)(i_q+1)} \cdot \text{sgn}(K', L') d_{j_p}^0 \dots d_{j_{r+1}}^0 d_{j_r}^2 \dots d_{j_1}^2(u_n) \otimes d_{i_q}^2 d_{i_{q-1}}^1 \dots d_{i_1}^1(u_n),$$

$$(K, L) = (i_q < \dots < i_1; 1 = j_p < \dots < j_1) \quad \text{and}$$

$$(K', L') = (1 < i_{q-1} < \dots < i_1; j_1 + 1 < \dots < j_{(r)} + 1 < j_p + j_{(r)} < \dots < j_{r+1} + j_{(r)})$$

with  $j_{(k)} = j_1 + \dots + j_k$  are unshuffles of the set  $\{1, \dots, n\}$ .

**Proposition 2.** *The map defined by formula (4.2) is a chain map.*

*Proof.* The proof is straightforward.  $\square$

Note that the components of the first summand of (4.2) together with ones of the second for  $(r, i_q) = (0, 1)$  agree with the components of the diagonal of the standard cube  $I^n$  [19], [9].

Using Proposition 1 formula (4.2) can be rewritten in the following combinatorial form (compare (6.3)):

$$(4.3) \quad \Delta_F(0, 1, \dots, n] =$$

$$\Sigma(-1)^{\epsilon_1} 0, 1, \dots, j_1][j_1, \dots, j_2][j_2, \dots, j_3] \dots [j_p, \dots, n] \otimes j_1, j_2, \dots, j_p, n] +$$

$$\Sigma(-1)^{\epsilon_2} j_r, \dots, j_{r+1}][j_{r+1}, \dots, j_{r+2}] \dots [j_p, \dots, n][0, 1, \dots, j_1][j_1, \dots, j_2] \dots [j_{r-1}, \dots, j_r] \otimes$$

$$j_t, \dots, j_p, n][0, j_1, \dots, j_r, \dots, j_t],$$

where the last tensor factor for a fixed  $r$  varies from  $j_{r+1}, \dots, j_p, n][0, j_1, \dots, j_{r+1}]$  to  $n][0, j_1, \dots, j_p, n]$ .



For example, formulas (4.2) and (4.3) for  $F_2$  and  $F_3$  read:

$$\Delta_F(u_2) = (d_1^0 d_2^0 \otimes Id + Id \otimes d_1^1 d_2^1 - d_1^0 \otimes d_2^1 + d_2^0 \otimes d_1^1 + (d_2^0 + d_1^2) \otimes d_2^2) (u_2 \otimes u_2);$$

$$\begin{aligned} \Delta_F(u_3) = & (d_1^0 d_2^0 d_3^0 \otimes Id + Id \otimes d_1^1 d_2^1 d_3^1 + d_1^0 \otimes d_2^1 d_3^1 - d_2^0 \otimes d_1^1 d_3^1 + d_3^0 \otimes d_1^1 d_2^1 + \\ & d_1^0 d_2^0 \otimes d_3^1 - d_1^0 d_3^0 \otimes d_2^1 + d_2^0 d_3^0 \otimes d_1^1 - \\ & (d_2^0 + d_1^2) \otimes d_2^2 d_3^1 + (d_3^0 + d_2^2) \otimes d_2^2 d_2^1 - (d_2^0 d_3^0 + d_2^0 d_1^2) \otimes (d_2^2 - d_3^2) + \\ & d_2^0 d_2^2 \otimes d_3^2) (u_3 \otimes u_3) \end{aligned}$$

and

$$\Delta_F(012) = 0[01][12] \otimes 012 + 012 \otimes 2[02] - 0[012] \otimes 02 + 01[12] \otimes 12[01] + (01[12] + 12[01]) \otimes 2[012];$$

$$\begin{aligned} \Delta_F(0123) = & 0[01][12][23] \otimes 0123 + 0123 \otimes 3[03] + 0[0123] \otimes 03 - 01[123] \otimes 13[01] + \\ & 012[23] \otimes 23[02] + 01[12][23] \otimes 123[01] + 0[01][123] \otimes 013 - 0[012][23] \otimes 023 - \\ & (01[123] + 123[01]) \otimes 3[013] + (012[23] + 23[012]) \otimes 3[023] - \\ & (01[12][23] + 12[23][01]) \otimes (23[012] - 3[0123]) + \\ & 23[01][12] \otimes 3[0123]. \end{aligned}$$

The diagonal  $\Delta_F$  is compatible with the AW diagonal of the standard simplex  $\Delta^n$  under the cellular map  $\varphi : F_n \rightarrow \Delta^n$ . To see this it is also convenient to represent  $\varphi$  combinatorially as:

$$i_{s_t}, \dots, i_{s_{t+1}}][i_{s_{t+1}}, \dots, i_{s_{t+2}}] \dots [i_{s_{k-1}}, \dots, i_{s_k}, n][0, i_1, \dots, i_{s_1}][i_{s_1}, \dots, i_{s_2}] \dots [i_{s_{t-1}}, \dots, i_{s_t}] \xrightarrow{\varphi} (i_{s_t}, \dots, i_{s_{t+1}}).$$

In particular, the faces  $0[0, 1, \dots, n]$  and  $n[0, 1, \dots, n]$  of  $F_n$ , i.e.  $d_1^0$  and  $d_n^2$ , go to the minimal and maximal vertices  $0 \in \Delta^n$  and  $n \in \Delta^n$  respectively (see Figure 3).

Note that the diagonal  $\Delta_F$  on  $C_*(F_n)$  is not coassociative, and hence, the product on  $C^*(F_n)$  is not associative, however since the acyclicity of  $F_n$  there exists an  $A_\infty$ -algebra structure on  $C^*(F_n)$  (see Subsection 6.2 below).

**4.1. The diagonal on an  $F_n$ -set.** Given an  $F_n$ -set  $\mathcal{CH}$ , define the chain complex  $(C_*(\mathcal{CH}), d)$  of  $\mathcal{CH}$  with coefficients in  $\mathbb{k}$  and with differential  $d_r : C_r(\mathcal{CH}) \rightarrow C_{r-1}(\mathcal{CH})$  given by

$$d_r = \bigoplus_{\substack{r=m+n \\ m, n \geq 0}} d_{m,n}, \quad d_{m,n} = \sum_{i=1}^{m+n} (-1)^i (d_i^0 - d_i^1) + \sum_{i=2}^m (-1)^{(i-1)(m+n)} d_i^2.$$

The *normalized chain complex* of  $\mathcal{CH}$  is  $(C_*^\odot(\mathcal{CH}), d) = (C_*(\mathcal{CH}), d)/D$ , where  $D$  is the subcomplex of  $C_*(\mathcal{CH})$  formed by degeneracies. Then apply (4.2) to make  $C_*^\odot(\mathcal{CH})$  as a dg *coalgebra*. In particular, given the singular  $F_n$ -set  $\text{Sing}^F Y$ , we get the dg coalgebra  $C_*^\odot(\text{Sing}^F Y)$  denoted by  $(C_*^\odot(Y), d)$ . Then the cellular composition

$$I^k \times F_m \times I^\ell \xrightarrow{1 \times \phi \times 1} I^k \times I^m \times I^\ell = I^{k+m+\ell} \xrightarrow{\psi} \Delta^{k+m+\ell}$$

induces a chain map  $C_*(Y) \rightarrow C_*^\odot(Y)$  to obtain the following

**Proposition 3.** *There are the natural isomorphisms of the homologies*

$$H_*(Y) \approx H_*^\odot(Y) = H_*(C_*^\odot(Y), d)$$

*and the cohomology algebras*

$$H^*(Y) \approx H_\odot^*(Y) = H^*(C_\odot^*(Y), d).$$

## 5. TRUNCATING TWISTING FUNCTIONS AND BITWISTED CARTESIAN PRODUCTS

**5.1. The Cartier-Hochschild set  $\Lambda X$ .** Given a 1-reduced simplicial set  $(X, \partial_i, s_i)$ , i.e.  $X = \{X_0 = X_1 = \{*\}, X_2, X_3, \dots\}$ , recall the definition of a *truncating twisting function* [9]:

**Definition 2.** *Let  $X$  be a 1-reduced simplicial set and  $Q$  be a monoidal cubical set. A sequence of functions  $\tau = \{\tau_n : X_n \rightarrow Q_{n-1}\}_{n \geq 1}$  of degree  $-1$  is a truncating twisting function if it satisfies:*

$$\begin{aligned} \tau(x) &= e, & x &\in X_1, \\ d_i^0 \tau(x) &= \tau \partial_{i+1} \dots \partial_n(x) \cdot \tau \partial_0 \dots \partial_{i-1}(x), & x &\in X_n, \quad 1 \leq i < n, \\ d_i^1 \tau(x) &= \tau \partial_i(x), & x &\in X_n, \quad 1 \leq i < n, \\ \eta_n \tau(x) &= \tau s_n(x), & x &\in X_n, \quad n \geq 1. \end{aligned}$$

A useful characterization of a truncating twisting function is that the monoidal map  $f : \Omega X \rightarrow Q$  defined by  $f(\bar{x}_1 \dots \bar{x}_k) = \tau(x_1) \dots \tau(x_k)$  is a map of cubical sets, where  $\Omega X$  is the monoidal cubical set constructed in [9] such that  $\Omega X$  is related with  $X$  by the universal truncating twisting function  $\tau_U : X \rightarrow \Omega X$ ,  $x \mapsto \bar{x}$ .

**Definition 3.** *Let  $X = \{X_m, \partial_i, s_i\}$  be a 1-reduced simplicial set,  $Q$  be a monoidal cubical set, and  $L = \{L_n, d_i^\epsilon, \eta_i\}$  be a  $Q$ -bimodule via  $Q \times L \rightarrow L$  and  $L \times Q \rightarrow L$ . Let  $\tau = \{\tau_k : X_k \rightarrow Q_{k-1}\}_{k \geq 1}$  be a truncating twisting function. The bitwisted Cartesian product  $X_{\tau \times \tau} L$  is the bigraded set*

$$X_{\tau \times \tau} L = X \times L / \sim,$$

where  $(s_m(x), y) \sim (x, \eta_1(y))$ ,  $(x, y) \in X_m \times L_n$ , and endowed with the face  $d_i^0, d_i^1, d_i^2$  and degeneracy  $\eta_i$  operators defined by

$$\begin{aligned} d_i^0(x, y) &= \begin{cases} (\partial_1 \dots \partial_m(x), \tau(x) \cdot y), & i = 1, \\ (\partial_i \dots \partial_m(x), \tau \partial_0 \dots \partial_{i-2}(x) \cdot y), & 1 < i \leq m, \\ (x, d_{i-m}^0(y)), & m < i \leq m+n, \end{cases} \\ d_i^1(x, y) &= \begin{cases} (\partial_{i-1}(x), y), & 1 \leq i \leq m, \\ (x, d_{i-m}^1(y)), & m < i \leq m+n, \end{cases} \\ d_i^2(x, y) &= (\partial_0 \dots \partial_{i-1}(x), y \cdot \tau \partial_{i+1} \dots \partial_m(x)), \quad 1 \leq i \leq m, \\ \eta_i(x, y) &= (x, \eta_i(y)), \quad 1 \leq i \leq n+1. \end{aligned} \tag{5.1}$$

Using Proposition 1 it is easy to verify that  $(X_{\tau \times \tau} L, d_i^0, d_i^1, d_i^2, \eta_i)$  forms an  $F_n$ -set. In particular,

$$d_1^2(x, y) = (\partial_0(x), y \cdot \tau \partial_2 \dots \partial_m(x)) = (\partial_0(x), y \cdot e) = (\partial_0(x), y) = d_1^1(x, y).$$

Take in the above definition  $Q = L = \Omega X$ ,  $\tau = \tau_U$ , and the bimodule actions inducing by the monoidal product on the cubical set  $\Omega X$  to give the following

**Definition 4.** Given a 1-reduced simplicial set  $X$ , the Cartier-Hochschild set  $\mathbf{\Lambda}X$  is the bitwisted Cartesian product  $X_{\tau_U \times \tau_U} \mathbf{\Omega}X$  endowed with the  $F_n$ -set structure via (5.1).

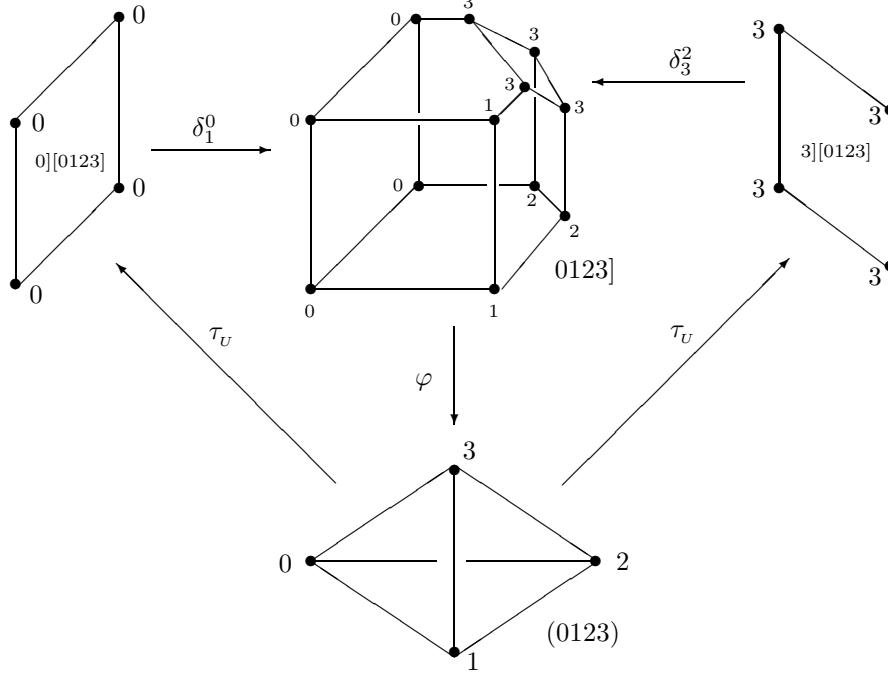


Figure 3. The two-fold interpretation of the universal truncating twisting function  $\tau_U$ .

**Remark 1.** 1. Note that in the Cartier-Hochschild set  $\mathbf{\Lambda}X$  one implies the identity  $d_1^0(x, e) = d_m^2(x, e) = \tau_U(x) = \bar{x}$  for any simplex  $x \in X_m$ .  
2. The operators  $d^0, d^1$  just subject to the defining identities of a cubical set.

## 6. THE BITWISTED CARTESIAN MODEL FOR THE FREE LOOP FIBRATION

Let  $\Omega Y \rightarrow \Lambda Y \xrightarrow{\xi} Y$  be the free loop space fibration on a topological space  $Y$ . Let  $\text{Sing}^1 Y \subset \text{Sing} Y$  be the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard  $n$ -simplex  $\Delta^n$  to the base point  $y$  of  $Y$ . Denote by  $C_*(Y)$  the quotient coalgebra  $C_*(\text{Sing}^1 Y)/C_{>0}(\text{Sing} y)$ , the chain complex of  $Y$ . Let  $\text{Sing}^I \Omega Y$  be the singular cubical set of  $\Omega Y$  and  $C_*^\square(\Omega Y)$  be the (normalized) chain complex of  $\text{Sing}^I \Omega Y$ . Then Adams' map  $\omega_* : \Omega C_*(Y) = C_*(\Omega \text{Sing}^1 Y) \rightarrow C_*^\square(\Omega Y)$  is realized by a monoidal cubical map  $\omega : \Omega \text{Sing}^1 Y \rightarrow \text{Sing}^I \Omega Y$  [9]. Obviously, we have the short sequence of singular  $F_n$ -sets

$$\text{Sing}^F \Omega Y \longrightarrow \text{Sing}^F \Lambda Y \xrightarrow{\xi_\#} \text{Sing}^F Y.$$

On the other hand, there is the short sequence of sets

$$\mathbf{\Omega} \text{Sing}^1 Y \rightarrow \mathbf{\Lambda} \text{Sing}^1 Y \xrightarrow{p} \text{Sing}^1 Y$$

where the maps are the natural inclusion and projection respectively. Let

$$\iota : \text{Sing}^I \Omega Y \rightarrow \text{Sing}^F \Omega Y$$

be the inclusion determined via the identification  $(\text{Sing}^I \Omega Y)_* = (\text{Sing}^F \Omega Y)^{0,*}$ .

We have the following

**Theorem 2.** *Let  $\Omega Y \rightarrow \Lambda Y \xrightarrow{\xi} Y$  be the free loop fibration.*

(i) *There are natural maps of sets  $\tilde{\varphi}, \Upsilon, \tilde{\omega}$  such that*

$$(6.1) \quad \begin{array}{ccccc} \text{Sing}^F \Omega Y & \longrightarrow & \text{Sing}^F \Lambda Y & \xrightarrow{\xi_{\#}} & \text{Sing}^F Y \\ \tilde{\omega} \uparrow & & \Upsilon \uparrow & & \tilde{\varphi} \uparrow \\ \Omega \text{Sing}^1 Y & \longrightarrow & \Lambda \text{Sing}^1 Y & \xrightarrow{p} & \text{Sing}^1 Y, \end{array}$$

$\tilde{\varphi} : \text{Sing}^1 Y \rightarrow \text{Sing}^F Y$  is a map induced by the composition  $I^k \times F_m \times I^\ell \xrightarrow{1 \times \phi \times 1} I^{k+m+\ell} \xrightarrow{\psi} \Delta^{k+m+\ell}$ , while  $\Upsilon$  is a map of  $F_n$ -sets, and  $\tilde{\omega} = \iota \circ \omega$ ; moreover, these maps are homotopy equivalences whenever  $Y$  is simply connected.

(ii) *The chain complex  $C_*^\odot(\Lambda \text{Sing}^1 Y)$  of the bitwisted Cartesian product*

$$\Lambda \text{Sing}^1 Y = \text{Sing}^1 Y_{\tau_U \times \tau_U} \Omega \text{Sing}^1 Y$$

*coincides with the Cartier complex  $\Lambda C_*(Y)$  of the chain coalgebra  $C_*(Y)$ .*

*Proof.* To define the map  $\Upsilon : \Lambda \text{Sing}^1 Y \rightarrow \text{Sing}^F \Lambda Y$ , it is convenient to apply a homotopically equivalent description of the free loop fibration  $\xi$  (cf. [12]). Namely,  $\xi$  is thought of as the associated fibre bundle with the universal bundle  $G \rightarrow EG \rightarrow BG$  via the conjugation action  $G \times G \rightarrow G$ ,  $(a, b) \rightarrow a^{-1}ba$ , where  $G$  has the homotopy type of  $\Omega Y$  and let  $\pi : EG \times G \rightarrow (EG \times G)/\sim = \Lambda BG$  be the quotient map.

Fix a section  $s : Y \rightarrow \Lambda BG$ . Choose its factorization  $s : Y \xrightarrow{s'} EG \times G \xrightarrow{\pi} \Lambda BG$  ( $s'$  does not need to be continuous). Fix a homotopy  $\chi_k : I^k \times I \rightarrow I^k$  contracting  $I^k$  at the minimal vertex. Let  $\rho : F_m \rightarrow I^m = I^{m-1} \times I$  be a cellular projection obtained by dilatation of the face  $d_m^2(F_m)$  to  $d_1^1(I^m)$ . Given  $(\sigma_m, \sigma'_n) \in (\Lambda \text{Sing}^1 Y)^{m,n}$ , let  $\Upsilon(\sigma_m, \sigma'_n) \in (\text{Sing}^F Y)^{m,n}$  be a map  $f : F_m \times I^n \rightarrow \Lambda BG$  defined as follows: Consider the compositions

$$\begin{aligned} \zeta : F_m \times I^n &\xrightarrow{\rho \times 1} I^{m-1} \times I \times I^n \xrightarrow{p_1} I^{m-1} \times I^n \xrightarrow{\tilde{\omega}(\bar{\sigma}_m, \sigma'_n)} G, \\ h : F_m \times I^n &\xrightarrow{\rho \times 1} I^{m-1} \times I \times I^n \xrightarrow{p_2} I^{m-1} \times I \xrightarrow{\chi^{m-1}} I^{m-1} \xrightarrow{\tilde{\omega}(\bar{\sigma}_m)} G, \end{aligned}$$

where  $p_1$  and  $p_2$  are canonical projections. Given  $(u, v) \in F_m \times I^n$ , let

$$f(u, v) = \pi(x_u \cdot h(u, v), y_u \cdot \zeta(u, v)),$$

where  $(x_u, y_u) = (\varphi \circ \sigma_m \circ s')(u) \in EG \times G$ . Since  $\varphi \circ \sigma_m \circ s : F_m \rightarrow \Lambda BG$  is continuous, it is easy to verify that  $f$  will be continuous as well. The relation  $(xa, ab) \sim (x, ba)$  in  $(EG \times G)/\sim$  guarantees the compatibility  $d_i^2(\sigma_m, \sigma'_n)$  with  $d_i^2(f)$  under  $\Upsilon$ . Thus, for  $m = 0, 1$  the map  $\Upsilon(\sigma_m, e) : F_m \rightarrow \Lambda BG$  is constant to the base point  $\lambda_0 = \pi(x_0, e)$ , where  $e$  denotes the unit of the monoid  $\text{Sing}^I \Omega Y$  too, and  $x_0$  is the base point of  $EG$ . On the other hand, we have  $\Upsilon(\sigma_0, \sigma'_n) : I^n \xrightarrow{\tilde{\omega}(\sigma'_n)} G \hookrightarrow \Lambda BG$  for  $n \geq 0$ .

The proof of  $\Upsilon$  being a homotopy equivalence (after the geometric realization) immediately follows, for example, from the comparison of the standard spectral sequences for  $\xi$  and  $p$  using the fact that  $\tilde{\omega}_*$  and  $\tilde{\varphi}_*$  are homology isomorphisms.

(ii) The proof is straightforward as the proof of the identification isomorphism  $C_*^\square(\mathbf{P} \operatorname{Sing}^1 Y) = \Omega(C_*(Y); C_*(Y))$  in [9]. Only we remark that the differential of  $\Lambda C_*(Y)$  differs from the one of the above acyclic cobar construction by the component  $\theta_2$  that agrees with the component  $d^2$  of the differential  $d$  of the chain complex  $C_*^\odot(\mathbf{A} \operatorname{Sing}^1 Y)$  under the required identification here.  $\square$

Thus, by passing to chain complexes in diagram (6.1) we obtain the following comultiplicative model of the free loop fibration  $\xi$  formed by dgc's.

**Theorem 3.** *For the free loop space fibration  $\Omega Y \rightarrow \Lambda Y \xrightarrow{\xi} Y$  there is a comultiplicative model formed by dgc's which is natural in  $Y$  :*

$$(6.2) \quad \begin{array}{ccccc} C_*^\odot(\Omega Y) & \longrightarrow & C_*^\odot(\Lambda Y) & \xrightarrow{\xi_*} & C_*^\odot(Y) \\ \tilde{\omega}_* \uparrow & & \Upsilon_* \uparrow & & \tilde{\varphi}_* \uparrow \\ \Omega C_*(Y) & \longrightarrow & \Lambda C_*(Y) & \xrightarrow{p_*} & C_*(Y). \end{array}$$

Obviously, one obtains the dual statement for cochain complexes involving the Hochschild dg algebra  $\Lambda C^*(Y)$ .

**6.1. The canonical homotopy  $G$ -algebra structure on  $C^*(X)$ .** Recall that there exists a canonical hga structure  $\{E_{k,1} : C^*(X)^{\otimes k} \otimes C^*(X) \rightarrow C^*(X)\}_{k \geq 0}$  on the simplicial cochain algebra  $C^*(X)$  [3], [7], [9] which, in particular, defines an associative multiplication  $\mu_E$  on the bar construction  $BC^*(X)$ . It is convenient to view these operations as the dual of the cooperations  $E^{k,1}$  on the simplicial chain coalgebra  $C_*(X)$ . In turn, these cooperations can be obtained by a combinatorial analysis of the diagonal of  $I^n$  :

$$(6.3) \quad \Delta[0, 1, \dots, n+1] = \Sigma(-1)^\epsilon [0, 1, \dots, j_1][j_1, \dots, j_2][j_2, \dots, j_3] \dots [j_p, \dots, n+1] \otimes [0, j_1, j_2, \dots, j_p, n+1].$$

where the summands  $[01 \dots n+1] \otimes [0, n+1]$  and  $[01][12][23] \dots [n, n+1] \otimes [01 \dots n+1]$  form the primitive part of the diagonal.

Regard the blocks of natural numbers above as faces of the standard  $(n+1)$ -simplex and discard the expression of the form  $[j, j+1]$  to obtain Baues' formula for a 1-reduced simplicial set  $X$  and a generator  $\sigma \in C_{n+1}(X)$  :

$$(6.4) \quad E^{k,1}(\sigma) = \Sigma(-1)^\epsilon (\sigma(0, 1, \dots, j_{s_1}) \otimes \sigma(j_{s_2}, \dots, j_{s_3}) \otimes \dots \otimes \sigma(j_{s_k}, \dots, n+1)) \otimes \sigma(0, j_1, j_2, \dots, j_p, n+1),$$

where  $\sigma(i_1, \dots, i_r)$  denotes the suitable face of  $\sigma$  (i.e.  $[i_1, \dots, i_r] = \tau_U \sigma(i_1, \dots, i_r)$ ) and  $k \leq p$ .

Now let  $\lambda_E$  denote an inducing multiplication by (4.2) on the complex  $\Lambda C^*(X)$ . Apply formulas (4.3), (6.3), (6.4) to write down  $\lambda_E$  in terms of operations  $E_{k,1}$ . Namely, given two elements  $u \otimes [\bar{a}_1 | \dots | \bar{a}_n]$  and  $v \otimes [\bar{b}_1 | \dots | \bar{b}_m]$  in  $\Lambda C^*(X)$ , we get

$$\begin{aligned}
(6.5) \quad \lambda_E \left( (u \otimes [\bar{a}_1 | \cdots | \bar{a}_m]) \otimes (v \otimes [\bar{b}_1 | \cdots | \bar{b}_n]) \right) = \\
\sum_{p=0}^m (-1)^{\varepsilon_1} u \cdot E_{p,1}(a_1, \dots, a_p; v) \otimes \mu_E([\bar{a}_{p+1} | \cdots | \bar{a}_m] \otimes [\bar{b}_1 | \cdots | \bar{b}_n]) + \\
\sum_{0 \leq i \leq j \leq k \leq m} (-1)^{\varepsilon_2} E_{m+i+1-k,1}(a_{k+1}, \dots, a_m, u, a_1, \dots, a_i; b_n) \cdot E_{j-i,1}(a_{i+1}, \dots, a_j; v) \otimes \\
\mu_E([\bar{a}_{j+1} | \cdots | \bar{a}_k] \otimes [\bar{b}_1 | \cdots | \bar{b}_{n-1}]), \\
\varepsilon_1 = \epsilon_p^a + (\epsilon_p^a + \epsilon_m^a)|v|, \\
\varepsilon_2 = \epsilon_m^a + (|u| + \epsilon_k^a)(\epsilon_k^a + \epsilon_m^a) + (|v| + \epsilon_{n-1}^b)(|b_n| + 1) + (\epsilon_j^a + \epsilon_k^a)(|v| + 1), \\
\text{where } \epsilon_r^x = |x_1| + \cdots + |x_r| + r.
\end{aligned}$$

**Remark 2.** The first summand component of (6.5) agrees with (13) in [9] up to signs: The sign component  $\epsilon_p^a$  correctly shown above is omitted in [9].

Thus, formula (6.5) gives the product on  $\Lambda C^*(Y)$  by setting  $X = \text{Sing}^1 Y$ . For example, for  $m = n = 1$  we have (up to signs):

$$\begin{aligned}
(6.6) \quad \lambda_E \left( (u \otimes [\bar{a}]) \otimes (v \otimes [\bar{b}]) \right) = u \cdot v \otimes ([\bar{a} | \bar{b}] + [\bar{b} | \bar{a}] + E_{1,1}(a; b)) + \\
u \cdot E_{1,1}(a; v) \otimes [\bar{b}] + E_{1,1}(u; b) \cdot v \otimes [\bar{a}] + \\
(E_{1,1}(u; b) \cdot E_{1,1}(a; v) + E_{2,1}(u, a; b) \cdot v + E_{2,1}(a, u; b) \cdot v) \otimes [\bar{ }].
\end{aligned}$$

Note that [9]  $E_{1,1}$  is in fact Steenrod's original definition of the cochain  $\smile_1$  operation. It satisfies the following Hirsch formula

$$c \smile_1 a \cdot b = (c \smile_1 a) \cdot b + (-1)^{|a|(|c|+1)} a \cdot (c \smile_1 b)$$

saying that  $\smile_1$  is the left derivation with respect to the  $\cdot$  (cup) product on  $C^*(Y)$ . On the other hand, the map  $-\smile_1 c : C^*(Y) \rightarrow C^*(Y)$  is a derivation only up to homotopy with the operation  $E_{2,1}$  serving as a suitable homotopy:

$$\begin{aligned}
dE_{2,1}(a, b; c) = E_{2,1}(da, b; c) - (-1)^{|a|} E_{2,1}(a, db; c) + (-1)^{|a|+|b|} E_{2,1}(a, b; dc) - \\
(-1)^{|a|} a \cdot b \smile_1 c + (-1)^{|a|+|b|+|b||c|} (a \smile_1 c) \cdot b + (-1)^{|a|} a \cdot (b \smile_1 c),
\end{aligned}$$

the Hirsch formula up to homotopy. In the next subsection we point out the other role of the operation  $E_{2,1}$ .

**6.2. Interaction between the Stasheff and Gerstenhaber higher order operations on  $\Lambda C^*(Y)$ .** Since the diagonal  $\Delta_F$  is not coassociative, the multiplication  $\lambda_E$  on the Hochschild chain complex  $\Lambda C^*(Y)$  is not associative; but Theorem 2(ii) and the acyclicity of  $F_m \times I^n$  guarantees the existence of an  $A_\infty$ -algebra structure on  $\Lambda C^*(Y)$ . Since formula (4.2), it is expected to construct this structure precisely. Indeed, denoting  $A = \Lambda C^*(Y)$  and  $x, y, z \in A$  with  $x = u \otimes [\bar{ }], y = v \otimes [\bar{ }], z = 1 \otimes [\bar{b}]$ ,  $du = dv = db = 0$ , apply (6.6) to obtain the following  $\lambda_E$  products in  $A$  (up to sign):

$$\begin{aligned}
(xy)z &= u \cdot v \otimes [\bar{b}] + (u \cdot v \smile_1 v) \otimes [\bar{ }] \quad \text{and} \\
x(yz) &= u \cdot v \otimes [\bar{b}] + ((u \smile_1 b) \cdot v + u \cdot (v \smile_1 b)) \otimes [\bar{ }].
\end{aligned}$$

From the Hirsch formula up to homotopy immediately follows the equality

$$d(E_{2,1}(u, v; b) \otimes [\bar{ }]) = (xy)z - x(yz).$$

Consequently, an operation  $\varphi^3 : A^{\otimes 3} \rightarrow A$  can be chosen with

$$\varphi^3(u \otimes [], v \otimes [], 1 \otimes [\bar{b}]) = E_{2,1}(u, v; b) \otimes [].$$

Note that the operation  $E_{2,1}$  is unavoidable on the *simplicial* cochain algebra  $C^*(Y)$  (i.e. there exists no strict Hirsch formula for the both sides simultaneously), and, consequently, its Hochschild chain algebra becomes a natural occurring and simplest example in which there is a non-trivial  $A_\infty$ -algebra structure on the chain level.

## 7. PROOF OF THEOREM 1

Recall the definition of an hga  $(A, d, \cdot, \{E_{p,q}\}_{p \geq 0, q=0,1})$ , in general [6], [7], [9]. Given  $k \geq 1$ , we have the following defining identities for it:

$$(7.1) \quad \begin{aligned} dE_{k,1}(a_1, \dots, a_k; b) &= \sum_{i=1}^k (-1)^{\epsilon_{i-1}^a} E_{k,1}(a_1, \dots, da_i, \dots, a_k; b) \\ &\quad + (-1)^{\epsilon_k^a} E_{k,1}(a_1, \dots, a_k; db) \\ &\quad + \sum_{i=1}^{k-1} (-1)^{\epsilon_i^a} E_{k-1,1}(a_1, \dots, a_i a_{i+1}, \dots, a_k; b) \\ &\quad + (-1)^{\epsilon_k^a + |a_k| |b|} E_{k-1,1}(a_1, \dots, a_{k-1}; b) \cdot a_k \\ &\quad + (-1)^{|a_1|} a_1 \cdot E_{k-1,1}(a_2, \dots, a_k; b), \end{aligned}$$

$$(7.2) \quad \begin{aligned} E_{k,1}(a_1, \dots, a_k; b \cdot c) \\ = \sum_{i=0}^k (-1)^{|b|(\epsilon_i^a + \epsilon_k^a)} E_{i,1}(a_1, \dots, a_i; b) \cdot E_{k-i,1}(a_{i+1}, \dots, a_k; c) \end{aligned}$$

and

$$(7.3) \quad \begin{aligned} \sum_{\substack{k_1 + \dots + k_p = k \\ 1 \leq p \leq k+1}} (-1)^\epsilon E_{p,1} \left( E_{k_1, \ell_1}(a_1, \dots, a_{k_1}; b'_1), \dots, E_{k_p, \ell_p}(a_{k-k_p+1}, \dots, a_k; b'_p); c \right) \\ = E_{k,1}(a_1, \dots, a_k; E_{\ell,1}(b_1, \dots, b_\ell; c)), \\ b'_i \in \{1, b_1, \dots, b_\ell\}, \quad \epsilon = \sum_{i=1}^p (|b'_i| + 1)(\epsilon_{k_i}^a + \epsilon_k^a), b'_i \neq 1. \end{aligned}$$

A *morphism*  $f : A \rightarrow A'$  between two hga's is a dga map  $f$  commuting with all  $E_{p,q}$ . Obviously, formula (6.5) has a sense for the Hochschild chain complex of an arbitrary hga and then  $f$  induces a dga map  $\Lambda f : \Lambda A \rightarrow \Lambda A'$ , and, consequently, an algebra map  $HH(f) : HH(A) \rightarrow HH(A')$ . However, in the lemma below  $f$  is not necessarily an hga map, nevertheless it induces an algebra map on the Hochschild homologies.

We need the following lemma in which for simplicity the subscripts are removed for the operations  $E_{k,1}$ .

**Lemma 1.** *Let  $A, A'$  be two  $\mathbb{k}$ -free hga's and let  $f : A \rightarrow A'$  be a dga map such that there is a sequence of maps  $s = \{s_{k,1} : A^{\otimes k+1} \rightarrow A'\}_{k \geq 1}$  with*

$$fE_{k,1} - E'_{k,1}f^{\otimes k+1} = -(s_{k,1} + s_{k-1,1})D - d's_{k,1}, \quad D : A^{\otimes k+1} \rightarrow A^{\otimes k+1} \oplus A^{\otimes k},$$

where  $D(a_1, \dots, a_k; b) = \sum_{i=1}^k (-1)^{\epsilon_{i-1}} (a_1, \dots, da_i, \dots, a_k; b) + (-1)^{\epsilon_k} (a_1, \dots, a_k; db) + \sum_{i=1}^{k-1} (-1)^{\epsilon_i} (a_1, \dots, a_i a_{i+1}, \dots, a_k; b)$ , and

$$(7.4) \quad s(a_1, \dots, a_k; b \cdot c) = \sum_{i=0}^{n-1} (-1)^{(|b|+1)\epsilon_i + |b|(\epsilon_k^a + 1)} E'(fa_1, \dots, fa_i; fb) \cdot s(a_{i+1}, \dots, a_k; c) \\ + \sum_{j=1}^n (-1)^{|b|(\epsilon_j^a + \epsilon_k^a)} s(a_1, \dots, a_j; b) \cdot fE(a_{j+1}, \dots, a_k; c),$$

and let  $\mathbf{s} : BA \otimes BA \rightarrow BA'$  be the extension of a map  $\bar{s} : BA \otimes BA \rightarrow A'$ ,

$$\bar{s}([\bar{a}_1 | \dots | \bar{a}_n] \otimes [\bar{b}_1 | \dots | \bar{b}_m]) = \begin{cases} s(a_1 \otimes \dots \otimes a_n \otimes b_1), & m = 1 \\ 0, & \text{otherwise,} \end{cases}$$

as a  $(\mu_{E'} \circ (Bf \otimes Bf), Bf \circ \mu_E)$ -coderivation. Then a map

$$\chi : \Lambda A \otimes \Lambda A \rightarrow \Lambda A'$$

defined for  $(u \otimes [\bar{a}_1 | \dots | \bar{a}_n]) \otimes (v \otimes [\bar{b}_1 | \dots | \bar{b}_m]) \in \Lambda A \otimes \Lambda A$  by

$$\chi((u \otimes [\bar{a}_1 | \dots | \bar{a}_m]) \otimes (v \otimes [\bar{b}_1 | \dots | \bar{b}_n])) \\ = \sum_{p=0}^m (-1)^{\nu_1 + |u| + \epsilon_p^a + |v|} u \cdot fE(a_1, \dots, a_p; v) \otimes \mathbf{s}([\bar{a}_{p+1} | \dots | \bar{a}_m] \otimes [\bar{b}_1 | \dots | \bar{b}_n]) \\ + \sum_{p=0}^m (-1)^{\nu_1 + |u|} u \cdot s(a_1, \dots, a_p; v) \\ \otimes (\mu_{E'} \circ (Bf \otimes Bf))([\bar{a}_{p+1} | \dots | \bar{a}_m] \otimes [\bar{b}_1 | \dots | \bar{b}_n]) \\ + \sum_{\substack{0 \leq i \leq j \\ \leq k \leq m}} (-1)^{\nu_2 + |u| + |b_n| + |v| + \epsilon_j^a + \epsilon_k^a + \epsilon_m^a} fE(a_{k+1}, \dots, a_m, u, a_1, \dots, a_i; b_n) \\ \cdot fE(a_{i+1}, \dots, a_j; v) \otimes \mathbf{s}([\bar{a}_{j+1} | \dots | \bar{a}_k] \otimes [\bar{b}_1 | \dots | \bar{b}_{n-1}]) \\ + \sum_{\substack{0 \leq i \leq j \\ \leq k \leq m}} (-1)^{\nu_2 + |u| + |b_n| + \epsilon_i^a + \epsilon_k^a + \epsilon_m^a} fE(a_{k+1}, \dots, a_m, u, a_1, \dots, a_i; b_n) \\ \cdot s(a_{i+1}, \dots, a_j; v) \otimes (\mu_{E'} \circ (Bf \otimes Bf))([\bar{a}_{j+1} | \dots | \bar{a}_k] \otimes [\bar{b}_1 | \dots | \bar{b}_{n-1}]) \\ + \sum_{\substack{0 \leq i \leq j \\ \leq k \leq m}} (-1)^{\nu_2} s(a_{k+1}, \dots, a_m, u, a_1, \dots, a_i; b_n) \cdot E'(fa_{i+1}, \dots, fa_j; fv) \\ \otimes (\mu_{E'} \circ (Bf \otimes Bf))([\bar{a}_{j+1} | \dots | \bar{a}_k] \otimes [\bar{b}_1 | \dots | \bar{b}_{n-1}])$$

is a chain homotopy between  $\Lambda f \circ \lambda_E$  and  $\lambda_{E'} \circ (\Lambda f \otimes \Lambda f)$ .

*Proof.* The proof is straightforward using equality (7.4).  $\square$

**Remark 3.** This lemma emphasizes a role of the explicit formula for the product  $\lambda_E$  in the following way: Since it has a general form  $\sum (A_1 \cdot A_2 \otimes A_3)$ , the chain homotopy  $\chi$  admits to be of the form  $\chi = A_1 \cdot A_2 \otimes s_3 + A_1 \cdot s_2 \otimes A'_3 + s_1 \cdot A'_2 \otimes A'_3$ , i.e. a standard derivation extension of maps  $s_i$  which are thought to be related with the factors by  $A_i - A'_i = ds_i + s_i d$ .



Using Lemma 1 we have the following comparison proposition.

**Proposition 4.** *Let  $f : A \rightarrow A'$  be as in Lemma 1. Then  $f$  induces an algebra map*

$$HH(f) : HH(A) \rightarrow HH(A')$$

*and when  $H(f) : H(A) \rightarrow H(A')$  is an isomorphism, so is  $HH(f)$ .*

Now let fix on  $H$  the trivial hga structure, i.e.  $\{E_{p,q}\} = \{E_{0,1}, E_{1,0}\}$ , while on  $C^*(Y; \mathbb{k})$  the canonical hga structure  $\{E_{p,q}\}$  mentioned in the previous section. Then we construct an auxiliary hga  $(RH, d)$  with dg algebra maps

$$H \xleftarrow{\rho} RH \xrightarrow{f} C^*(Y; \mathbb{k})$$

such that both maps satisfy the hypotheses of Lemma 1 and are cohomology isomorphisms. Then we can apply Proposition 4 to obtain algebra isomorphisms

$$HH_*(H) \xleftarrow{HH(\rho)} HH_*(RH) \xrightarrow{HH(f)} HH_*(C^*(Y; \mathbb{k})).$$

Since the product  $\lambda_E$  on  $\Lambda(H, 0, \cdot, \{E_{0,1}, E_{1,0}\})$  coincides with the standard shuffle product, one gets algebra isomorphisms [10]

$$HH_*(H, 0, \cdot, \{E_{0,1}, E_{1,0}\}) \approx S(U) \otimes \Lambda(s^{-1}U) \approx H(Y; \mathbb{k}) \otimes H(\Omega Y; \mathbb{k});$$

on the other hand, from the previous section we have an algebra isomorphism

$$HH_*(C^*(Y; \mathbb{k}), d, \cdot, \{E_{p,q}\}) \approx H^*(\Lambda Y; \mathbb{k}).$$

Consequently, Theorem 1 follows.

Thus it remains to define the hga  $RH$  and maps  $\rho, f$  mentioned above: Indeed, consider a bigraded multiplicative resolution  $\rho : (RH, d) \rightarrow H$  of  $H$  ([16], [18]) such that  $R^*H^* = T(V^{*,*})$  with  $V^{*,*} = V^{0,*} \oplus \mathcal{E}^{<0,*} \oplus \mathcal{T}^{-2r,*}$ ,  $r \geq 1$ ,  $V^{0,*} \approx U^*$ ,  $\mathcal{E}^{-n,*} = \{\mathcal{E}_{k,1}^{-n,*}\}_{1 \leq k \leq n}$  with  $\mathcal{E}_{k,1}^{-n,*}$  spanned on the set of expressions  $E_{k,1}(a_1, \dots, a_k; b)$ ,  $a_r \in R^{-i_r}H^*$ ,  $b \in V^{-j,*}$ ,  $n = \sum_{r=1}^k i_r + j$ , unless  $E_{1,1}(a; a)$ ,  $a \in V^{0,*}$ , and subjected to relations (7.3), while  $\mathcal{T}^{-2(n-1),*}$ ,  $n \geq 2$ , is spanned on the set of expressions  $a_1 \smile_2 a_2 \smile_2 \dots \smile_2 a_n$  with  $a_i \in V^{0,*}$ ,  $a_i \smile_2 a_j = a_j \smile_2 a_i$ , and  $a_i \neq a_j$  for  $i \neq j$ ; the differential  $d$  is defined: On  $V^{0,*}$  by  $dV^{0,*} = 0$ ; on  $\mathcal{E}$  by formula (7.1), and on  $\mathcal{T}$  by

$$(7.5) \quad d(a_1 \smile_2 \dots \smile_2 a_n) = \sum_{(i;j)} (a_{i_1} \smile_2 \dots \smile_2 a_{i_k}) \smile_1 (a_{j_1} \smile_2 \dots \smile_2 a_{j_\ell})$$

where the summation is over unshuffles  $(i;j) = (i_1 < \dots < i_k; j_1 < \dots < j_\ell)$  of  $\underline{n}$  and  $\smile_1$  denotes  $E_{1,1}$ . In particular,  $dE_{1,1}(a; b) = dE_{1,1}(b; a) = ab - ba$  and  $d(a \smile_2 b) = E_{1,1}(a; b) + E_{1,1}(b; a)$  for  $a, b \in V^{0,*}$ . It is straightforward to check that  $H(R^{<0}H^*, d) = 0$  (or see the argument for an analogous resolution, denoted by  $R_\delta H$ , in [18]). Set  $E_{1,1}(a; a) = 0$  for  $a \in V^{0,*}$  and extend the operations  $E_{k,1}(a_1, \dots, a_k; b)$ ,  $k \geq 1$ , by formula (7.2) on  $RH$  for any  $b \in RH$ . Thus,  $\rho$  becomes an hga map too. Consequently,  $HH(\rho)$  is an isomorphism by Proposition 4.

Since  $Sq_1[z] = 0$ , denote by  $\gamma(z; z) \in C^{2n-2}(Y; \mathbb{k})$  a cochain such that

$$(7.6) \quad d\gamma(z; z) = z \smile_1 z.$$

Next define a dga map  $f : (RH, d) \rightarrow C^*(Y; \mathbb{k})$  as follows. First define it on  $V$  and then extend multiplicatively. On  $V^{0,*}$  by choosing cocycles  $f^0 : U^* \rightarrow C^*(Y; \mathbb{k})$ ; on  $\mathcal{T}$ : by  $f(a_1 \smile_2 a_2) = f^0 a_1 \smile_2 f^0 a_2$ , where  $\smile_2$  denotes Steenrod's cochain operation, and extend inductively for  $a_1 \smile_2 \dots \smile_2 a_n$ ,  $n \geq 3$ ; such an extension has no obstructions, since a cocycle in  $C^*(Y)$  written by cochain operations in

distinct variables is cohomologous to zero; on  $\mathcal{E}$  : for  $E_{k,1}(a_1, \dots, a_k; b)$ ,  $k \geq 1$ , with  $a_i \in RH$  and  $b \in V^{0,*}$  set

$$fE_{k,1}(a_1, \dots, a_k; b) = \begin{cases} E_{2,1}(fa_1, fa_2; fb) \\ -fa_1 \cdot \gamma(fa_2; fb) - \gamma(fa_1; fb) \cdot fa_2, & k = 2 \\ E_{k,1}(fa_1, \dots, fa_k; fb), & k \neq 2; \end{cases}$$

for  $b = E_{\ell,1}(b_1, \dots, b_\ell; c)$  and  $b' = E_{\ell-1,1}(b_1, \dots, b_{\ell-1}; c)$ ,  $b'' = E_{\ell-1,1}(b_2, \dots, b_\ell; c)$  set

$$fE_{k,1}(a_1, \dots, a_k; b) = \begin{cases} E_{k,1}(fa_1, \dots, fa_k; fb) \\ +(-1)^k E_{k-1,1}(fa_1, \dots, fa_{k-1}; fb_1) \cdot \gamma(fa_k; fc) \\ +(-1)^k E_{k-1,1}(fa_1, \dots, fa_{k-1}; fc) \cdot \gamma(fa_k; fb_1) \\ -\gamma(fa_1; fb_1) \cdot E_{k-1,1}(fa_2, \dots, fa_k; fc) \\ -\gamma(fa_1; fc) \cdot E_{k-1,1}(fa_2, \dots, fa_k; fb_1), & \ell = 1 \\ E_{k,1}(fa_1, \dots, fa_k; fb) \\ +(-1)^k E_{k-1,1}(fa_1, \dots, fa_{k-1}; fb') \cdot \gamma(fa_k; fb_\ell) \\ -\gamma(fa_1; fb_1) \cdot fE_{k-1,1}(a_2, \dots, a_k; b''), & \ell \geq 2, \end{cases}$$

where we assume  $\gamma(a; b) = 0$  unless  $a = b$  with  $a \in V^{0,*}$  in which case  $\gamma$  is defined by (7.6); and, finally, for  $b \in \mathcal{T}$ , set

$$fE_{k,1}(a_1, \dots, a_k; b) = E_{k,1}(fa_1, \dots, fa_k; fb).$$

Define maps  $s_{k,1} : RH^{\otimes k+1} \rightarrow C^*(Y; \mathbb{k})$ ,  $k \geq 1$ , of degree  $-1$  first for  $a_1 \otimes \dots \otimes a_k \otimes b \in RH^{\otimes k} \otimes V$  by

$$s_{k,1}(a_1 \otimes \dots \otimes a_k \otimes b) = \begin{cases} \gamma(b; b), & k = 1, a_1 = b \in V^{0,*} \\ 0, & \text{otherwise} \end{cases}$$

and then extend them on whole  $RH^{\otimes k+1}$  by formula (7.4). It is immediate to verify that  $f$  and  $s = \{s_{k,1}\}_{k \geq 1}$  satisfy the hypotheses of Proposition 4 and, consequently,  $HH(f)$  is an isomorphism.  $\square$

**Example 1.** Let  $A = \mathbb{k}[x, y]$ ,  $|x| = |y| = 2$ ,  $B = T(\bar{x}, \bar{y}, z)/\{\bar{x}^2, \bar{y}^2\}$ ,  $|\bar{x}| = |\bar{y}| = |z| = 1$ ,  $dz = \bar{x}\bar{y} + \bar{y}\bar{x}$ . Take  $(A, 0) \otimes (B, d)$  and set  $h(z) = x$  to obtain the dga  $C = (A \otimes B, d^\otimes + h)$ . It is easy to see that the spectral sequence of  $C$  is collapsed, and, consequently, its  $E_\infty$ -term is isomorphic as algebras with the  $E_\infty$ -term of the Serre spectral sequence of the free loop fibration with the base  $Y = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ . However,  $H^*(C)$  is isomorphic only additively with  $H^*(\Lambda Y)$ .

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